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Daniel Aalto: *Boundedness of maximal operators and oscillation of functions in metric measure spaces*; Helsinki University of Technology Institute of Mathematics Research Reports A585 (2010).

Abstract: In this dissertation the action of maximal operators and the properties of oscillating functions are studied in the context of doubling measure spaces. The work consists of four articles, in which boundedness of maximal operators is studied in several function spaces and different aspects of the oscillation of functions are considered. In particular, new characterizations for the BMO and the weak L^∞ are obtained.

AMS subject classifications: 42B25, 43A85, 46E35

Keywords: doubling measure, maximal functions, discrete convolution, BMO, John-Nirenberg inequality, rearrangements

Daniel Aalto: *Maksimaalioperaattorien rajoittuneisuudesta ja funktioiden heilahtelusta metrisessä avaruudessa*

Tiivistelmä: Väitöskirjassa tutkitaan maksimaalioperaattoreita ja heilahtelevien funktioiden ominaisuuksia tuplaavassa metrisessä avaruudessa. Työ koostuu neljästä julkaisusta, joissa tarkastellaan maksimaalioperaattoreiden rajoittuneisuutta eri funktioavaruuksissa sekä funktioiden heilahtelua useista eri näkökulmista. Erityisesti saadaan uusia luonnehdintoja avaruuksille BMO ja heikko L^∞.

Avainsanat: tuplaava mitta, maksimaalifunktiot, diskreetti konvoluutio, BMO, John-Nirenbergin lemma, uudelleenjärjestelyt

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Bern, March 2010,

Daniel Aalto

List of included articles

This dissertation consists of an overview and of the following publications:

[A] D. Aalto, J. Kinnunen, Maximal functions in Sobolev spaces, Sobolev Spaces in Mathematics I, International Mathematical Series, Vol. 8 Maz'ya, Vladimir (Ed.), 25-68, Springer, 2008

[B] D. Aalto, J. Kinnunen, The discrete maximal operator in metric spaces, to appear in J. Anal. Math.

[C] D. Aalto, Weak L^∞ and BMO in metric spaces, arXiv:0910.1207 [math.MG].

[D] D. Aalto, L. Berkovits, O. E. Maasalo, H. Yue, John-Nirenberg lemmas for doubling measures, arXiv0910.1228 [math.FA].

Author's contribution

The author is responsible for a substantial part of preparing all of the articles [A-D]. The results have been presented in talks given at the Centre de Recerca Matemàtica in Barcelona, at the Institute of Mathematics of the Polish Academy of Sciences in Warsaw and at the Universities of Jyväskylä, Bern, Turku and Helsinki.

The articles [A] and [B] have been prepared in Helsinki University of Technology during 2005-2008 in collaboration with Juha Kinnunen; the article [C] was done in the University of Turku in 2009; the article [D] was contributed by the author while visiting at Universität Bern and when at Helsinki University of Technology and University of Turku during 2008-2009.

Contents

1 Analysis on metric spaces

Analysis on metric spaces refers to a research field where many classical results of harmonic analysis in Euclidean spaces and the theory of partial differential equations are extended to a more general geometry. The framework was set in the seminal work of Coifman and Weiss in 1971 [13]. For an introduction to analysis on metric spaces we refer to [22]. In this overview, we first introduce the general setting and then describe the key results of the thesis.

1.1 The doubling condition

A metric space consists of a set X and a distance d. A ball with center $x \in X$ and of radius $r > 0$ is the set

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

It is tacitly assumed that whenever B is a ball it has some fixed center and radius. For $\lambda > 0$, we write λB for the ball with the same center as B but of λ times the radius of B.

A metric space is doubling if there exists a constant N so that every ball $B(x, 2r)$ can be covered by at most N balls of radius r. In contrast, a doubling measure is a Borel measure in a metric space (X, d) for which

$$0 < \mu(B(x, 2r)) \leq c\mu(B(x, r)) < \infty$$

for all $x \in X$ and all $r > 0$. The smallest $c > 0$ satisfying the above inequality is called c_μ, the doubling constant of the measure μ. A metric space with a doubling measure is called a doubling metric measure space.

The two different doubling conditions are closely related. Indeed, any doubling metric measure space is a doubling metric space. Moreover, in every complete doubling metric space there exists a doubling measure [33], see also [42]. In this thesis, however, we assume that the doubling measure is given a priori.

1.2 Coverings

In every doubling metric space there holds a Vitali covering theorem according to which of any uniformly bounded family of balls it is possible to extract a countable subfamily of pairwise disjoint balls whose 5-dilates cover the original family. An important consequence of the Vitali theorem is Lebesgue's differentiation theorem. Indeed, every locally integrable function f can be written in terms of mean value integrals,

$$\lim_{r \to 0} \fint_{B(x,r)} f \, \mathrm{d}\mu = f(x)$$

at almost every point x. Here we use notation

$$\fint_S f \, \mathrm{d}\mu = \frac{1}{\mu(S)} \int_S f \, \mathrm{d}\mu$$

for the mean value integral over a set S.

We shall also use a Whitney covering theorem. Indeed, for an open subset $\Omega \subset X$ with non-empty complement, it is possible to find a covering by balls so that their overlap is uniformly bounded and that the radii of the balls are proportional to the distance to the complement. In the Euclidean space the theorem is usually stated in terms of dyadic cubes.

In addition to the Vitali and Whitney covering theorems we will use several variants of the Calderón-Zygmund lemmas. The most geometric of them states that almost every point of a bounded measurable set E can be covered by a countable family of balls B_i, $i = 1, 2, \dots$, such that the total mass of the balls is comparable to the measure of E and so that for each ball B_i the part intersecting E is comparable in measure to the part intersecting the complement of E. Other versions of the lemma also involve a function and the role of the set E is played by distribution sets of a suitable maximal function.

1.3 Sobolev spaces on metric spaces

The Sobolev spaces on metric spaces can be defined in several ways. Here we adopt the definition by Shanmugalingam, which is based on the notion of upper gradients; see [39]. Indeed, a Borel measurable function g is the upper gradient of a measurable function u defined on X if

$$|u(x) - u(y)| \le \int_\gamma g \, \mathrm{d}s,$$

for all $x, y \in X$ and for all rectifiable paths γ joining points x and y. If u is not finite at some point, then we require that $\int_\gamma g \, \mathrm{d}s$ is infinite for all rectifiable paths γ that start at the point; see [23]. To obtain the completeness of the Sobolev space, we extend the concept of the upper gradient. Indeed, we define a p-weak upper gradient of u by requiring the above inequality for p-almost every path (i.e. for every rectifiable path of the space X except for a path family of zero p-modulus), $p \ge 1$; see [32]. The modulus of a path family Γ is defined by

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X \rho^p \, \mathrm{d}\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$, for which

$$\int_\gamma \rho \, \mathrm{d}s \ge 1$$

for every locally rectifiable $\gamma \in \Gamma$. If u has a p-weak upper gradient that is p-integrable, then it has a minimal p-weak upper gradient g_u; see [19]. Also,

u is absolutely continuous along p-almost every path. We define a norm by setting

$$\|u\|^p_{N^{1,p}(X)} = \|u\|^p_{L^p(X)} + \|g_u\|^p_{L^p(X)},$$

where g_u is the minimal upper gradient. The Newtonian space is the collection of the p-integrable functions on X that has finite norm. We identify two elements in $N^{1,p}(X)$ with each other if the norm of their difference is zero. The Newtonian space is a Banach space. For more details and proofs, see [39] and [7]. The definition of the Newtonian space for an open subdomain Ω of X is similar to that of the whole space. Moreover, we are interested in the boundary behaviour of the functions and would like to define the functions with zero boundary values. We say that a function u defined on an open set belongs to the Sobolev space with zero boundary values, if there exists a function v defined on the whole space such that u and v coincide on the domain of u, and v equals zero on the complement of the domain except for a set of capacity zero; see [26].

1.4 Poincaré inequality

Although the definition of the Newtonian space makes sense formally in every metric space X, the Newtonian space coincides with the Lebesgue space of p-integrable functions if there are no rectifiable paths in X. We assume that the space X supports a Poincaré inequality, which guarantees the existence of a multitude of rectifiable paths. Indeed, a metric space supports a weak $(1,p)$-Poincaré inequality if there exist constants $c>0$ and $\tau \geq 1$ such that for all balls $B(x,r)$, for all locally integrable functions u on X, and for all p-weak upper gradients g of u,

$$⨍_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq cr \left(⨍_{B(x,\tau r)} g^p \, \mathrm{d}\mu \right)^{1/p}. \tag{1}$$

By Hölder's inequality a $(1,p)$-Poincaré inequality implies a $(1,p+\delta)$-Poincaré inequality for all $\delta > 0$. Moreover, $(1,p)$-Poincaré inequality implies a $(1,p-\epsilon)$-Poincaré inequality for some $\epsilon > 0$; see [25].

The Poincaré inequality has several noteworthy geometric and analytic consequences. Indeed, a complete doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality is quasiconvex, i.e. every pair of points can be joined by a curve with length comparable to the distance of the points, the Lipschitz functions are dense in the Newtonian space $N^{1,p}(\Omega)$ for any open set Ω of X, and finally, the Sobolev embedding theorems hold; see [39]. Next we state the embedding results more precisely.

1.5 Sobolev embeddings

The Sobolev embedding theorems refer to a series of inequalities valid for a specific range of p with respect to the dimension of the measure. By iterating the doubling condition we have

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq c\left(\frac{r}{R}\right)^Q,$$

for any given $B(x,R)$, $y \in B(x,R)$ and $r < R$ with the constant c depending only on the doubling constant of the measure. We call the number $Q = \log_2(c_\mu)$ the dimension of the measure. The first Sobolev inequality is for small values of p. Indeed, if $1 < p < Q$ and $1 \leq \kappa \leq Q/(Q-p)$, then there is $c = c(p, \kappa, c_\mu) > 0$ such that

$$\left(\fint_{B(z,r)} |u - u_{B(z,r)}|^{\kappa p}\, \mathrm{d}\mu\right)^{1/\kappa p} \leq cr\left(\fint_{B(z,5\tau r)} g^p\, \mathrm{d}\mu\right)^{1/p}$$

for every p-weak upper gradient of u. Here τ is the constant in the inequality 1. If $p = Q$, then there are constants $c_1, c_2 > 0$ depending only on the doubling constant such that

$$\fint_{B(z,r)} \exp\left(\frac{c_1\mu(B(z,r))|u - u_{B(z,r)}|}{r\|g\|_{L^p(B(z,5\tau r))}}\right)\, \mathrm{d}\mu \leq c_2$$

for every p-weak upper gradient g of u. This borderline case is often referred to as the Trudinger inequality. If $p > Q$, then there is a constant $c = c(p, c_\mu) > 0$ such that

$$|u(x) - u(y)| \leq cr^{Q/p} d(x,y)^{1-Q/p}\left(\fint_{B(z,5\tau r)} g^p\, \mathrm{d}\mu\right)^{1/p}$$

for μ almost every $x, y \in B(z,r)$ and every p-weak upper gradient g of u. In other words, every Sobolev function can be redefined in a set of measure zero to become locally Hölder continuous; for proofs see [20] and [37]. In addition, when $1 < p \leq Q$, the Sobolev functions can be approximated by Hölder continuous functions in Lusin's sense; see [A] for the precise statement.

1.6 Hardy-Littlewood maximal function

The Hardy-Littlewood maximal function of a locally integrable function f is defined by

$$Mf(x) = \sup \fint_{B(y,r)} |f|\, \mathrm{d}\mu$$

where the supremum is taken over all balls that contain x. The maximal function is lower semicontinuous and the maximal operator is sublinear. The maximal operator maps integrable functions to the weak L^1. Indeed, for all $\lambda > 0$ we have

$$\mu(\{x \in X : Mf(x) > \lambda\}) \leq \frac{c}{\lambda}\int_X |f|\, \mathrm{d}\mu,$$

with c depending only on the doubling constant of the measure μ. The maximal operator is bounded between L^p for $p > 1$. These mapping properties of the maximal operator are often referred to as the maximal function theorem. Observe that the maximal operator is not bounded in L^1 in general. In fact, in the Euclidean case, if Mf is integrable with respect to the Lebesgue measure, then $f = 0$. On the other hand, if a maximal function is finite at some point, then it is finite almost everywhere; see [43], [15] and [A].

In the Euclidean space, the Hardy-Littlewood maximal operator acts boundedly on Lipschitz and Hölder classes, provided the maximal function is not identically infinite; see [14], and [A]. In addition, the Hardy-Littlewood maximal operator is bounded on Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ for $p > 1$; see [27]. The boundedness of the Hardy-Littlewood maximal operator remains true also if the Sobolev space is defined over a subdomain of $\mathbb{R}^n$ by [29]. For an alternative proof see [21]. Moreover, the boundary values of a Sobolev function carry over to the maximal function in the Sobolev sense; see [30]. However, the maximal function of a differentiable function may have points of non-differentiability, although the weak derivatives are preserved.

In a doubling metric space the situation changes remarkably. Indeed, an example by Buckley in [10] shows that the maximal operator may map a Lipschitz function to a discontinuous one. Consequently, the maximal operator cannot be bounded in Lipschitz, Hölder, or Sobolev classes in all doubling measure spaces. However, if the metric measure space satisfies some extra conditions, the maximal operator is bounded in Lipschitz and Hölder classes [10] and similar conditions also guarantee the boundedness in Sobolev spaces [35], [36]. Kinnunen and Latvala took a different approach by constructing a discrete maximal operator similar to the Hardy-Littlewood maximal operator and showed that under the standard assumptions, i.e. X is a complete doubling measure space supporting the weak $(1,p)$-Poincaré inequality, the discrete maximal operator is bounded between Sobolev spaces; see [28]. They applied the result in showing that Sobolev functions have Lebesgue points outside a set of capacity zero; see also [8].

2 The discrete maximal operator

In this section we consider the discrete maximal operator and introduce a local version of the operator. The definition is based on discrete convolutions [34] and the global maximal operator was introduced in [28].

In the article [B], we define the local discrete maximal operator by four ingredients: coverings of the subdomain by a Whitney covering, partitions of unity, discrete convolutions and a supremum over the discrete convolutions. More precisely, let $\Omega \subset X$ be an open set of a doubling metric space X. If $\Omega = X$, we take a covering of Ω by balls $\{B(x_i, r_i)\}$ of equal radii, $r_i = r > 0$ for all $i = 1, 2, \dots$, so that the overlap of the dilated balls, $\{B(x_i, 6r_i)\}$, is uniformly bounded. This is the global case. The local case is when $X \setminus \Omega$ is not empty. Then we introduce a Whitney type covering of Ω at scale

$0 < t < 1$. It consists of balls $\{B(x_i, r_i)\}$ that cover Ω so that

$$\kappa_1 r_i \le t \operatorname{dist}(x, X \setminus \Omega) \le \kappa_2 r_i$$

and

$$\sum_{i=1}^{\infty} \chi_{B(x_i, 6r_i)}(x) \le N$$

for every $x \in \Omega$. Here distance of a point to any set $A \subset X$ is defined by

$$\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}$$

and χ_A is the characteristic function of the set A. In the preceeding inequalities the constants κ_1, κ_2, and N are independent of the scale t. Now it is straightforward to proceed. Indeed, given a covering $\{B(x_i, r_i)\}$ as in the previous step, we attach to each $B(x_i, r_i)$ a Lipschitz continuous function $\psi_i : X \to [0, \infty)$, with a Lipschitz constant comparable to r_i. Moreover, we require that

$$c\chi_{B(x_i,3r_i)}(x) \le \psi_i(x) \le \chi_{B(x_i,6r_i)}(x)$$

where c is independent of the scale t and the index i, and

$$\sum_{i=1}^{\infty} \psi_i(x) = 1$$

for every $x \in \Omega$. Now we are ready to define the discrete convolution. Given a locally integrable function f defined in Ω we write

$$f_t(x) = \sum_{i=1}^{\infty} f_{B(x_i,3r_i)} \psi_i(x)$$

for the discrete convolution at scale $0 < t < 1$, where $\{B(x_i, r_i)\}$ is a Whitney covering and $\{\psi_i\}$ forms a partition of unity as above. The discrete maximal function is now the supremum of discrete convolutions at all rational scales. We write

$$M^*_\Omega f(x) = \sup_{0<t<1} |f|_t(x)$$

for the discrete maximal function. Observe, that the maximal function depends on the chosen coverings and partitions of unity, but the estimates for the maximal operator are independent of these choices.

The discrete maximal function is lower semicontinuous and the maximal operator is sublinear. In addition, the discrete maximal function is comparable to the Hardy-Littlewood maximal function. To be more precise, we define the local Hardy-Littlewood maximal function, with $\sigma \ge 1$, by

$$M_{\sigma,\Omega} f(x) = \sup \fint_{B(x,r)} |f| \, d\mu,$$

where the supremum is taken over all radii r for which $\sigma r \le \operatorname{dist}(x, X \setminus \Omega)$. If $\Omega = X$, then there is no restriction for the radii. Now there exist constants $c > 1$ and $\sigma > 1$ so that

$$c^{-1} M_{\sigma,\Omega} f(x) \le M^*_\Omega f(x) \le c M_\Omega f(x)$$

at every point x of Ω, where f is any locally integrable function. Here the constant c depends only on the doubling constant of the measure and the parameters in the construction of the maximal operator. The pointwise equivalence implies at once the standard maximal function theorem, i.e. the boundedness in L^p for $1 < p \le \infty$ and the weak type inequality for $p = 1$.

The discrete maximal operator is also bounded between Newtonian spaces. Indeed, an essential ingredient in the arguments is the Poincaré inequality. Since it is combined with the maximal function theorem, the theorem of Keith and Zhong [25] is also employed. The first boundedness result of the discrete maximal operator was obtained by Kinnunen and Latvala [28] in the context of Hajłasz spaces defined globally in X. In an open subdomain the results are proven in [B], where the results are considered in the Newtonian setting. The essential difference in the arguments in [B] and [28] is not the choice of the definition of the Sobolev space. Also, the proof of the boundedness of the discrete maximal operator is rather similar in local and global cases. In contrast, there is a new question arising from the local problem: What happens to the boundary values of a given function? In the Euclidean case we know that the boundary values are preserved [30] and we show in [B] that the corresponding result generalizes; i.e. the difference of a Newtonian function and its discrete maximal function belongs to the Newtonian space with zero boundary values, $N^{1,p}_0(\Omega)$. In the proof we use Hardy's inequality which guarantees that the Newtonian function belongs to the Sobolev space of zero boundary values.

The preceding results span the whole range of $1 < p < \infty$. However, taking account of the Sobolev embedding theorems, it is natural to ask whether the larger spaces have similar behaviour with respect to the maximal operators. Indeed, in [B] it is shown that the Hölder continuity is preserved. The argument only uses the structure of the discrete maximal operator and the fact that the metric space is doubling.

In the borderline case there are several natural spaces to be considered. The Hardy-Littlewood maximal operator is bounded in the Zygmund class of exponentially integrable functions (see [16], [18]) and it acts boundedly in BMO, as well. The Euclidean argument in [11] is based on Coifman-Rochberg theorem (see [13], [44]), true also in the doubling measure spaces, and the John-Nirenberg inequality, and hence generalizes to metric spaces. The arguments can be adapted to the discrete maximal operator as well by using the pointwise equivalence of the maximal operators. In the Euclidean case there are a number of related results in spaces similar to BMO; see [5], [6], [31], and [40]. Next we focus in the intrinsic properties of function spaces defined by oscillations.

3 Oscillation of functions

Bennett, DeVore and Sharpley introduced in [5] the function space weak L^∞. The space is defined in terms of rearrangements and oscillations and they showed that the rearrangement invariant hull of BMO coincides with the weak L^∞, when the underlying space is a Euclidean cube with Lebesgue measure. This result clearly indicates that there is an intimate connection between the two spaces. There are also sharper forms of the Trudinger inequality, which show that the Sobolev space $W^{1,n}(\Omega)$ is embedded in the weak $L^\infty(\Omega)$; see [3] and [1] for even sharper results. Instead of studying the Sobolev embedding theorems, we focus on the function spaces as such.

3.1 Rearrangements and weak L^∞

The decreasing rearrangement of a measurable function f defined on X is the unique decreasing function $f^* : (0,\infty) \to [0,\mu(X)]$ equimeasurable to f, i.e.

$$\mu(\{x \in X : |f(x)| > \lambda\}) = |\{t > 0 : f^*(t) > \lambda\}|$$

for all $\lambda > 0$, and right continuous. Here we use the notation $|A|$ for the Lebesgue measure of the subset A of $\mathbb{R}^n$. Note that the equimeasurability condition alone does not determine the function uniquely. By definition f belongs to the weak L^∞, if f^* is finite everywhere and there exists $M \geq 0$ such that

$$\frac{1}{t}\int_0^t (f^*(s) - f^*(t))\ \mathrm{d}s \leq M$$

for all $t > 0$. Observe that essentially bounded functions are always contained in the weak L^∞ and the constant functions satisfy the defining inequality with constant $M = 0$.

We are interested in finding what is the relationship between weak L^∞ and BMO. Towards this end we look for a more geometric characterization of the former. Indeed, it is possible to completely avoid the rearrengements in the definition. In fact, the first result of [C] states that a function belongs to the weak L^∞ if and only if there exists $M \geq 0$ so that

$$\int_{\{x\in X:|f(x)|>\lambda\}} |f|\ \mathrm{d}\mu \leq (\lambda + M)\mu(\{x \in X : |f(x)| > \lambda\})$$

for every $\lambda > 0$ and $\mu(\{x \in X : |f(x)| > \lambda\})$ is not infinite for all $\lambda > 0$.

The proof of the characterization only uses the basic properties of decreasing rearrengements and a version of the Cavalieri principle according to which every measurable function f satisfies

$$\begin{aligned}&\int_{\{x\in X:|f(x)|>\lambda\}} |f|\ \mathrm{d}\mu \\ &= \int_\lambda^\infty \mu(\{x \in X : |f(x)| > s\})\ \mathrm{d}s + \lambda\mu(\{x \in X : |f(x)| > \lambda\})\end{aligned}$$

for every $\lambda > 0$. Since the Cavalieri principle holds for any measure space, the characterization applies in the same generality. A more detailed analysis of the definitions of weak L^∞ leads to yet another characterization. Indeed, by a variant of the Grönwall-Bellman inequality (see [4]), if a function f belongs to the weak L^∞, then there exist constants $c_1, c_2 > 0$ so that

$$\mu(\{x \in X : |f(x)| > \lambda_2\}) \le c_1 \mu(\{x \in X : |f(x)| > \lambda_1\}) \mathrm{e}^{c_2(\lambda_2 - \lambda_1)}$$

for all $\lambda_2 > \lambda_1 \ge 0$. And conversely, by Cavalieri principle, we see that the above global John-Nirenberg type inequality characterizes the space.

By virtue of the characterization, we see that functions in the weak L^∞ have singularities of exponential type at worst. To state this consequence more precisely, we say that a measurable function f belongs to the Zygmund class $L_{\exp}(X)$ if

$$\int_X (\exp(\lambda|f|) - 1) \, \mathrm{d}\mu < \infty$$

for some $\lambda \ge 0$, which may depend on the function f. The following theorem is implicit already in [5].

Theorem 1. *Let (X, μ) be a measure space. Then*

$$L^\infty(X) \subset L^\infty_w(X) \subset L^\infty(X) + L_{\exp}(X)$$

and the inclusions are strict if and only if there are measurable sets of arbitrarily small positive measure in X.

Proof. Let $f \in L^\infty_w(X)$ and let $\alpha > 0$ so that

$$\mu(\{x \in X : |f(x)| > \alpha\}) < \infty.$$

By Theorem 2.1 in [C] we have

$$\mu(\{x \in X : |f(x)| > s\}) \le A\mathrm{e}^{-Bs}$$

for all $s > \alpha$, where $A, B > 0$ and depend only on the function f and α. Let

$$g(x) = f\chi_{\{y \in X : |f(y)| \le \alpha\}}(x)$$

and $h = f - g$. Since $f = g + h$ and $g \in L^\infty(X)$ it suffices to show that $h \in L_{\exp}(X)$. Let $0 < \lambda < B$. By the Cavalieri principle and a change of variables we have

$$\begin{aligned}
&\int_X (\exp(\lambda|h|) - 1) \, \mathrm{d}\mu \ \le \ \int_{\{x \in X : |f(x)| > \alpha\}} \exp(\lambda|f|) \, \mathrm{d}\mu \\
&= \int_\alpha^\infty \mu(\{x \in X : |f(x)| > s\}) \lambda \mathrm{e}^{\lambda s} \, \mathrm{d}s + \mathrm{e}^{\lambda s} \mu(\{x \in X : |f(x)| > \alpha\}), \\
&\le A\lambda \int_\alpha^\infty \mathrm{e}^{(\lambda - B)s} \, \mathrm{d}s + \mathrm{e}^{\lambda s} \mu(\{x \in X : |f(x)| > \alpha\}),
\end{aligned}$$

which is finite and hence the latter inclusion is shown.

Suppose now that there are measurable sets of arbitrarily small positive measure in X and let us construct suitable functions to show the properness of the inclusions. By the assumption, we may choose a sequence of measurable sets such that $\mu(E_1) < 1$ and

$$\mu(E_{i+1}) < \frac{1}{4}\mu(E_i)$$

for $i = 1, 2, \dots$. Setting now

$$A_i = E_i \setminus \bigcup_{j=i+1}^{\infty} E_j$$

we get a pairwise disjoint sequence of measurable sets so that

$$\mu(A_{i+1}) < \frac{1}{2}\mu(A_i).$$

Let

$$f(x) = \sum_{k=1}^{\infty} 2^k \chi_{A_{2^k}}(x)$$

and

$$g(x) = \sum_{k=1}^{\infty} k\chi_{A_k}(x).$$

We claim that $g \in L^{\infty}_w(X)$ and $f \in L_{\exp}(X)$. First observe that for given $\lambda_2 > \lambda_1 > 0$ we have

$$\mu(\{x \in X : |g(x)| > \lambda_2\}) = \sum_{k>\lambda_2} \mu(A_k) \leq 2^{\lambda_1-\lambda_2+1}\mu(\{x \in X : |g(x)| > \lambda_1\})$$

by the construction and hence g belongs to the weak $L^{\infty}(X)$ but obviously it is not bounded. Since g is contained in $L_{\exp}(X)$ and $f \leq g$ in X, we see that f is exponentially integrable. But

$$\fint_{\{x\in X:|f(x)|>2^k+1\}} |f| \, \mathrm{d}\mu \geq 2^{k+1} = (2^k + 1) + (2^k - 1),$$

and hence f cannot belong to the weak $L^{\infty}(X)$.

Assume now that the measure of each measurable set in X is either zero or uniformly bounded from below. Then

$$L_{\exp}(X) \subset L^1(X) \subset L^{\infty}(X),$$

which proves the theorem. □

The functions constructed during the proof of the previous theorem serve also as examples to show that weak L^{∞} is usually not a vector space. Indeed, the functions g and $f - g$ are clearly contained in the weak L^{∞} but their sum is not. Hence, the weak L^{∞} is a vector space only when it coincides with L^{∞}.

3.2 John-Nirenberg lemma and BMO

Since the definition of BMO in 1961, the John-Nirenberg inequality has been one of the most important properties of BMO functions. Because functions in the weak L^∞ satisfy a global John-Nirenberg type inequality, it seems very natural to ask what is the relationship of BMO and weak L^∞. Bennett, DeVore, and Sharpley showed that, in the Euclidean space equipped with the Lebesgue measure, the rearrangement invariant hull of BMO coincides with the weak L^∞. This cannot happen in general, since there exists a non-doubling Radon measure μ (see [38]) in the real line so that the corresponding BMO space does not satisfy the John-Nirenberg inequality and consequently the weak L^∞ cannot contain all of the BMO. There is no general inclusion in the other direction either. Indeed, there exists a Radon measure in the real line so that not all the functions in the corresponding weak L^∞ can be rearranged to become a BMO function (see [C]).

In a doubling metric measure space, the relationship is clear. Indeed, the main result of [C] shows that $BMO(X)$ is contained in the weak $L^\infty(X)$. The argument is similar to the Euclidean case in [5]. The main difficulty is in finding a good substitute for the Euclidean Calderón-Zygmund covering lemma, which is based on dyadic cubes. With the metric version the argument generalizes and the same reasoning can be adapted to give a new characterization of BMO. Indeed, when $\rho > 1$, a locally integrable function f belongs to $BMO(X)$ if and only if there exists $M \geq 0$ such that

$$\int_{\{x\in B:|f(x)-f_B|>\lambda\}} |f - f_B| \, \mathrm{d}\mu \leq (\lambda + M)\mu(\{x \in \rho B : |f(x) - f_B| > \lambda\})$$

for every $\lambda > 0$ and every ball B contained in X.

If the metric measure space has some additional geometric structure, say (X, d) is geodesic, then the parameter ρ is superfluous. The reason is that if the doubling metric measure space is restricted to a ball of the space, then the resulting space is also a doubling metric measure space. This additional feature can be used now to ameliorate the Calderón-Zygmund type covering lemma so that the 5-balls are intersected with the original ball B_0 and these sets are comparable in size to the dilated balls.

Since the Euclidean space is geodesic, the above argument shows that for a doubling measure μ, the corresponding BMO space is contained in the weak L^∞ and we have a characterization for a BMO. The arguments can be generalized to some non-doubling measures as well in the spirit of [38].

3.3 Another John-Nirenberg lemma

We shall consider now a third approach to the oscillation of functions based on a joint work with Lauri Berkovits, Outi Elina Maasalo, and Hong Yue, as reported in [D]. John and Nirenberg defined in [24] a continuum of function spaces that lie between $L^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Indeed, a locally integrable function f belongs to John-Nirenberg space with exponent $1 \leq p < \infty$, we write $f \in JN_p(Q_0)$, where Q_0 is a cube, if there exists $A_f \geq 0$ so that

$$\sum_{i=1}^{\infty} \mu(Q_i) \left(\fint_{Q_i} |f - f_{Q_i}| \, \mathrm{d}\mu \right)^p \leq A_f^p$$

for all partitions $\{Q_i\}$ of Q_0. The definition coincides with the definition of BMO when p tends to infinity and with L^1, if $p = 1$. Observe that, as a consequence of Hölder's inequality, $L^p(Q_0)$ is contained in $JN_p(Q_0)$. Hence the natural counterpart of the exponential integrability of BMO functions is Marcinkiewicz estimate. Indeed, if $f \in JN_p(Q_0)$ with $1 < p < \infty$, then

$$|\{x \in Q_0 : |f(x) - f_{Q_0}| > \lambda\}| \leq C \left(\frac{A_f}{\lambda} \right)^p$$

for every $\lambda > 0$ and for some constant $C > 0$ independent of f. There are several proofs in the Euclidean case; see [24], [17] and [41]. We give yet another proof in [D].

To generalize the result in doubling metric measure spaces it is not even clear how to generalize the definition. One possibility is to use the dyadic construction by Christ [12]; see also [2]. The theorem, with the obvious changes in the definitions and statements, can be proved by the Euclidean proof in [D]. However, in a metric space the metric ball is more natural an object to study. Hence, we will also consider another definition in terms of balls. Indeed, we say that $f \in JN_p(B_0)$ for some metric ball B_0 and $1 < p < \infty$ if there exists a constant $A_f \geq 0$ such that

$$\sum_{i=1}^{\infty} \mu(B_i) \left(\fint_{B_i} |f - f_{B_i}| \, \mathrm{d}\mu \right)^p \leq A_f^p$$

for every collection of balls $\{B_i\}$ such that the balls $\{\frac{1}{5}B_i\}$ form a pairwise disjoint family contained in $11B_0$ with centers in B_0. As before $L^p(11B_0)$ is a subset of $JN_p(B_0)$. The proof is slightly longer and uses a maximal function argument. The underlying principle is the observation that if a pairwise disjoint family of balls in a doubling metric space is dilated, then the sum of the characteristic functions of the dilated balls is integrable to any power $1 < q < \infty$. In the Euclidean space, we can compare the definition by balls and by cubes. Indeed, if Q_0 is a cube contained in a ball B_0, then whenever $f \in JN_p(B_0)$ we also have $f \in JN_p(Q_0)$.

To show that the John-Nirenberg space is contained in the Marcinkiewicz space, we use Calderón-Zygmund decomposition that can be effectively iterated. A version that suits particularily well to our context is presented in [D]. Indeed, given a locally integrable function f and levels

$$\lambda_N > \lambda_{N-1} > \cdots > \lambda_1 \geq \frac{1}{\mu(B_0)} \int_{11B_0} |f| \, \mathrm{d}\mu,$$

we can choose N families of balls $\{B_i(\lambda_j)\}$, $j = 1, 2, \ldots, N$, so that each family is admissible in the definition of JN_p and f is at most λ_j almost everywhere in the complement of the union of the balls in the j^{th} family, the mean value integral of f over any of the balls $B_i(\lambda_j)$ is comparable to λ_j, and so that each $B_i(\lambda_j)$ is contained in some $5B_k(\lambda_{j-1})$ with $j = 2, \ldots, N$.

The lemma can be used to simplify the existing proofs (see [38], [9], [37]) of John-Nirenberg inequality in doubling measure spaces, as well.

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